Uncertainty in Portfolio Liquidation

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ABSTRACT

Portfolio valuation is usually done by valuing each security in the portfolio at its current market price. Effectively, it is assumed that portfolios may be immediately liquidated at prevailing prices without price concessions. This paper derives an analytical model without price concessions which makes three predictions. First, regardless of the number of stocks or initial wealth, the expected time required to liquidate a portfolio will always exceed the time required to form the portfolio. Second, increasing the amount invested increases both the portfolio-formation time and the portfolio-liquidation time, but the liquidation-time increase is always larger. Third, increasing the number of stocks considered for investment also dramatically increases the liquidation-time relative to the formation-time. Portfolios formed primarily with small-capitalization stocks will require much longer to liquidate than form because they combine the first and third effect above.

INTRODUCTION

Such common tasks as portfolio performance measurement and portfolio rebalancing depend critically on the way in which portfolio values are measured. Portfolio valuation is usually done by valuing each security in the portfolio at its current market price. Effectively, it is assumed that portfolios may be immediately liquidated at prevailing prices without price concessions. But should we expect that portfolios may be even relatively quickly liquidated?

In a market where the sequencing of security orders is random, it would appear at first glance that portfolio formation and portfolio liquidation times should be symmetrical, requiring roughly the same time to liquidate a portfolio as it takes to form it, in the absence of price concessions.\(^1\) Closer consideration, however, reveals that liquidation will take longer in such a market because you can only sell assets you own whereas during formation, you can buy whatever is available. It turns out that liquidation can take considerably longer - almost three times as long in most cases and sometimes much longer - than formation. What factors affect this time-to-liquidation and how can investors manage portfolio liquidation time?

This paper derives an analytical model without price concessions which makes three predictions. First, regardless of the number of stocks or initial wealth, the expected time required to liquidate a portfolio will always exceed the time required to form the portfolio. Second, increasing the amount invested increases both the portfolio-formation time and the portfolio-liquidation time, but the liquidation-time increase is always larger. Third, increasing the number of stocks considered for investment also dramatically increases the liquidation-time relative to the formation-time.

The remainder of this paper is organized as follows. Section 1 describes the theoretical model. Section 2 discusses the implications and the final section summarizes the findings and conclusions.

Theoretical Model

Consider a stock market in which \(K\) securities are traded. Assume that transactions take place sequentially and at fixed intervals of one unit of time. At any point in time, \(t\), only one of the \(K\) stocks will trade and, in the absence of price concessions, the probability \((p_t)\) is equal for all stocks. That is:

\[
p_t = \frac{1}{K}, \quad t = 1, 2, \ldots, K
\]

\(^1\)The term “no price concessions” means transacting at the current market quote for no more than the currently quoted volume. This is the classic liquidity trader role. Traders in search of immediacy may influence order arrival sequencing by offering to trade at better prices than the current quotes but, in this case, they are accepting a price concession by either paying more than the current market price (purchases) or receiving less (sales).
Also for simplicity and without any loss of generality, assume that whenever a transaction occurs, it is for one share and at a price of one unit of wealth.

An investor with \( W \) units of wealth wishes to invest his wealth fully in the \( K \) securities as quickly as possible and will buy every share as it becomes available until all wealth is invested. Since the stocks purchased are unconstrained and one transaction occurs after each interval of time consuming one unit of wealth, the total time to form the portfolio, \( T_f \), will always be the initial wealth, \( W \). The expected holdings and the expected time to liquidation are functions of the joint probability distribution of the possible stock holdings.

Let \( X_i \) be the number of shares of stock 1 in the portfolio after formation, \( X_2 \) be the number of shares of stock 2, and so on to \( X_K \). Since each purchase is independent of all other purchases and has exactly \( K \) possible results, the joint probability distribution function (p.d.f.) of \( (X_1, X_2, ..., X_K) \) is multi-nomial (Dudewicz and Mishra, 1988) and

\[
f_{X_1, X_2, ..., X_K}(x_1, x_2, ..., x_K) = \begin{cases} \frac{W!}{x_1! x_2! ... x_K!} p_1^{x_1} p_2^{x_2} ... p_K^{x_K}, & x_i = 0, 1, 2, ... , W, \quad 1 \leq i \leq K \\ 0, & \text{elsewhere} \end{cases}
\]  

(2)

The expected holding, \( E(X_i) \), of any stock, \( i \), is \( W/p_i \) or \( W/K \) since the marginal p.d.f. of \( X_i \) is binomial. On average, the investor expects to hold an equal amount of each stock in his portfolio at the end of the formation period.

When liquidating the portfolio, the expected time to liquidate the actual holding in a particular stock \( i \) is \( X_i/p_i \) or \( KX_i \), since, on average, one share of stock \( i \) will be sold every \( 1/p_i = K \) intervals. The expected time required to liquidate the whole portfolio, \( T_L \), will be a function of the maximum number of shares held. The expected time to liquidate the portfolio is:

\[
E(T_L) = \frac{E(\max[X_1, X_2, ..., X_K])}{p_1} = K \cdot E(\max[X_1, X_2, ..., X_K])
\]  

(3)

If \( Y = \max[X_1, X_2, ..., X_K] \), let \( G(y) \) be the probability that \( Y \leq y \), and \( g(y) \) be the probability that \( Y = y \). Then \( G(y) \) is given by the multiple summation

\[
G(y) = \Pr(\text{hol}(X_1 \leq y, X_2 \leq y, ..., X_K \leq y)) = \sum_{i=1}^{y} \sum_{j=0}^{y} \sum_{k=0}^{y} f_{X_1, X_2, ..., X_K}(x_1, x_2, ..., x_K)
\]  

(4)

and the p.d.f. of \( Y, g(y) \), may be found from:

\[
g(y) = G(y) - G(y^-)
\]  

(5)

where \( y^- \) means “approaches \( y \) from below.” The expected value of the maximum is given by:

\[
E(\max[X_1, X_2, ..., X_K]) = \sum_{r=y^{-}}^{y} r g(y)
\]  

(6)

Discussion

Equations (5) and (6) do not have tractable closed-form solutions for even moderately large \( K \) and \( W \), however, for reasonably small values (e.g., \( K \leq W \leq 200 \)), the probability distribution function in (5) and the expected value of (6) can be calculated numerically.\( ^2 \) Figure 1 shows the p.d.f. of the maximum share holding for \( K = 50 \) and \( W = 100 \). The expected maximum number of shares held is 5.77 per issue and it will require, on average, 288.75 units of time to liquidate the portfolio. Thus, while it takes 100 units of time to construct the portfolio, all else equal, it requires almost three times as long to liquidate it. Clearly, if the formation-time is anything but negligible, the assumption of unlimited instantaneous selling is inappropriate.

Simply stated, the model predicts that portfolios can be relatively quickly formed since all volume in qualified issues will be purchased as it arrives. Liquidation takes longer even though the arrival sequence remains random, since it is limited in two ways. First, one can only sell securities one owns and second, one can sell no more securities than are wanted at the bid price.

\(^2\) There are 192 distinctly different possible combinations of the \( X_i \)'s (independent of order) that must be considered given the 17 possible maximums produced when 20 purchases \( (W) \) are distributed over 5 stocks \( (K) \). This number jumps to 189,477,547 when \( W=100 \) and \( K=50 \).
Since the formation-time is a function of liquidity, portfolios constructed with less liquid stocks will require longer in the absence of price concessions to form and commensurately longer to liquidate. Since small firm stocks tend to fall most frequently into the illiquid category (see, Fowler, Rorke and Jog (1979, 1980), Keim (1989) and Griffiths (1993)), small-capitalization portfolios will manifest this effect the most.

Further, since a small-capitalization stock can only absorb a small amount of a large pool of capital, an investor must invest in many stocks to invest all available capital. Figure 2 illustrates the effects of increasing the number of stocks available and/or the wealth invested in the portfolio. As invested wealth increases, the ratio of the time to liquidate the portfolio over the time to form the portfolio decreases. This occurs because it is more likely that the investor holds the security which is wanted in the portfolio.

Investors with limited wealth to invest will be most affected by the liquidation delay shown in Figure 2 as they will have less wealth invested per stock. Portfolios with limited wealth achieve the lowest liquidation delay by investing in a lower number of highly liquid stocks.

**Summary and Conclusions**

This paper investigates the assumption, implicit in portfolio valuation techniques that mark-to-market, that portfolios may be immediately liquidated at prevailing prices without price concessions. An analytical model without price concessions is derived which makes three predictions. First, regardless of the number of stocks or initial wealth, the expected time required to liquidate a portfolio will always exceed the time required to form the portfolio. Second, increasing the amount invested increases both the portfolio-formation time and the portfolio-liquidation time, but the liquidation-time increase is always larger. Third, increasing the number of stocks considered for investment also dramatically increases the liquidation-time relative to the formation-time.

Portfolios formed primarily with small-capitalization stocks will require much longer to liquidate than form because they combine the first and third effect above. Portfolios with limited wealth achieve the lowest liquidation delay by investing in a lower number of highly liquid stocks.

Further research might investigate the size of the price concession needed to achieve timely liquidation; particularly for small-capitalization or limited wealth portfolios. Empirical testing would require order data that would include volumes and prices off the market.

**Figure 1.**

**P.D.F. of Maximum Share Holding, \( \text{Max}[X_1, X_2, \ldots, X_{50}] \)**

Initial Wealth of 100 Units and 50 Securities in Portfolio

- Expected Maximum Holding: 5.77 shares
- Expected Liquidation Time: 288.75 units
Probability distribution function of the largest share holding in the portfolio, \( \text{Max}[X_1,X_2,\ldots,X_{50}] \), when 100 units of wealth are invested across 50 securities. Transactions occur sequentially at fixed intervals of time and at any time, \( t \), only one share of one of the 50 stocks is available for purchase at a price of one unit of wealth (assuming no price concessions). The probability that any stock, \( i \), will be available at any time is the same for all stocks. Under these conditions, the largest single stock holding after portfolio formation is expected to be 5.77 shares. Forming this portfolio will require 100 units of time. The resulting portfolio is expected to require 288.75 units of time to liquidate.

**Figure 2.**

*Relative Time to Liquidate a Portfolio After Formation by Number of Securities in Portfolio (K)*

This figure shows the how long liquidation of a portfolio is expected to take relative to the time it takes to form the portfolio assuming that \( W \) units of wealth are invested across \( K \) stocks. Transactions occur sequentially at fixed intervals of time and at any time, \( t \), only one share of one of the \( K \) stocks is available for purchase or sale at a price of one unit of wealth (assuming no price concessions). The probability that any stock, \( i \), will be available at any time is the same for all stocks.

**REFERENCES**


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