An EOQ Model for Defective Items with Shortages

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ABSTRACT

An inventory management system for defective items with backordered shortages is explored, in which we assume that the quality of an ordered lot is not always 100% perfect, so a screening process to each product is conducted to split that lot into perfect and defective products. Meanwhile, the defective products include imperfect and scrap ones, which will be sold at a discount price and disposed of at a cost, respectively. This paper aims at finding the optimal order size and optimal backorder level for each cycle by minimizing a simpler objective of expected total cost per unit time, instead of utilizing the somewhat complex expected total profit per unit time in Eroglu and Ozdemir’s (2007) paper. Sequentially, we also make numerous previous models as special cases of this paper.

Keywords: EOQ; Defective; Shortages; Backorder

INTRODUCTION

Most existing economic order quantity models appearing in the literature have been developed by assuming that an ordered lot received at the beginning of a selling period is 100% perfect in quality. However, it may not be pertinent to real market environments, not only because of production processes, but also because of delivery processes or other unexpected factors, all of which might more or less damage the products’ quality.

Therefore, Porteus (1986) presented an EOQ model in association with the effect of defective items, where a probability that a production process would go out of control is hypothesized. Rosenblatt and Lee (1986) assumed that timing from the beginning of a production run to an uncontrollable process is an exponential distribution and the defective products can be reworked at that instant moment with an extra cost.

Salameh and Jaber (2000) extended the traditional EOQ problem by accounting for imperfect products in which an ordered lot is 100% screened, and that resulting imperfect products will be sold at a single batch when the screening process is completed. Later an error in their result toward the optimal order size was corrected by Cardenas-Barron (2000), showing that the denominator of the expression regarding the optimal order size should be multiplied by a factor of “2”. Likewise, Papachristos and Konstantaras (2006) further investigated the model with a proportional imperfect quality, which is a random variable. They expanded the models to the case that the defectives are withdrawn at the end of
planning horizon, other than at the end of the screening process. Recently, Eroglu and Ozdemir (2007) developed a related model with stock-out occurrence, during which shortages are backordered and the defective products, containing imperfect and scrap ones, will be sold at cheaper price and/or discarded at cost.

This paper aims at extending the Eroglu and Ozdemir’s (2007) model to the case that both defective and scrap rates are random variables, making our proposed model more realistic. Meanwhile, we recommend that an easier objective of minimizing the expected total cost per unit time is an equivalent substitute compared with the expected total profit per unit time in their model.

The remainder of this paper is organized as follows. Notation and assumptions are outlined in Section 2. Model formulation and objective function are presented in Section 3. In Section 4, we analyze the objective function and come to a conclusion about optimal values, along with some special cases and numerical example illustration. Finally, we close the paper with a conclusion and possible future research.

**NOTATION AND ASSUMPTIONS**

To be consistent with former research studies, we adopted the same notation as those in Eroglu and Ozdemir’s (2007) paper as follows.

- \( D \) = constant demand rate
- \( y \) = order size for each cycle (decision variable)
- \( w \) = maximal backorder level allowed (decision variable)
- \( c \) = unit purchasing cost
- \( s \) = unit selling price for perfect products
- \( v \) = unit selling price for imperfect products
- \( Cs \) = unit disposal cost for scrap products
- \( p \) = percentage of defective products in \( y \)
- \( \theta \) = percentage of scrap products in defective products
- \( f(p) \) = probability density function of \( p \)
- \( g(\theta) \) = probability density function of \( \theta \)
- \( E(\cdot) \) = expected value operator
- \( k \) = fixed ordering cost per order
- \( h \) = unit holding cost per unit time
- \( \pi \) = unit backorder cost per unit time
- \( x \) = screening rate per unit time
- \( d \) = unit screening cost
- \( t_1 \) = time to build up the backorder level \( w \)
- \( t_2 \) = time to eliminate the backorder level \( w \)
- \( t_3 \) = time to screen the order size \( y \)
- \( t \) = cycle length (decision variable)

In addition, we assume that a retailer replenishes a lot with size of \( y \) units at the beginning of each cycle, along with the fixed ordering cost \( k \) and unit purchasing cost \( c \). Each lot received is supposed to be 100% screened with the screening rate \( x \) and unit screening cost \( d \), and the percentage of defective products is an assumed random variable \( p \) with probability density function \( f(p) \). Afterwards, the
defective products are split into imperfect and scrap ones with the random variable $\theta$ and probability density function $g(\theta)$ for the scraps. Once the screening process is completed, the imperfect products are sold as a single batch with unit selling price $v$ and the scraps are disposed of at a unit cost $C_s$. The unit selling price for perfect products is the $s$ and the shortages are backordered with a cost $\pi$ per unit per unit time.

**MODEL FORMULATION**

The behavior of inventory level is illustrated in Fig. 1, where we let $t_1$ and $t_2$ be the time intervals to build up and eliminate the backorders $w$, respectively. The screening time interval is $t_3$, and $t$ is the cycle length. During the $t_2$ interval, the rate of the perfect products is $(1-p)x$, a part of which will meet the demand rate $D$ whereas the remaining is to eliminate the backorders with a rate of $(1-p)x-D=xA$, where $A=1-p-D/x$. At the end of the $t_3$ interval, the defective products are taken away from inventory. Thus, referring to the aforementioned assumptions and the Fig.1, we have the following results.

$$
t = \frac{(1-p)y}{D}
$$

implies

$$
E(t) = \frac{[1-E(p)]y}{D}
$$

(1)

$$
t_1 = \frac{w}{D}
$$

(2)

$$
t_2 = \frac{w}{xA} \quad \text{and} \quad t_2 = \frac{y-Z}{(1-p)x}
$$

Imply

$$
Z = y - \frac{(1-p)w}{A}
$$

(3)

$$
t_3 = \frac{y}{x} \quad \text{and} \quad t_3-t_2 = \frac{Z-Z_1-px}{D}
$$

Imply

$$
Z_1 = Ay - w
$$

(4)

Next, the objective function will be constructed. First, we calculate the total revenues over cycle period by $TR=s(1-p)y+v(1-\theta)py$, and then the expected total revenues per unit time is given as

$$
E(TR) = s\left[1-E(p)\right]y + v\left[1-E(\theta)\right]E(p)py
$$

$$
E(t) = \frac{[1-E(p)]y}{D}
$$

$$
= sD + \frac{vD[1-E(\theta)]E(p)}{1-E(p)}
$$

(5)

From (5), we find the expected total revenues per unit time is independent of $y$, and therefore, for the sake of simplicity, the minimization of the expected total cost per unit time is concerned rather than the maximization of the expected total profit per unit time in Ergüloğlu and Özdemir’s (2007) paper. The components of the total cost per cycle, denoted by $TC$, are listed below.

- **Purchasing cost** = $cy$
- **Ordering cost** = $k$
- **Screening cost** = $dy$
- **Disposal cost** = $C_s\theta py$
- **Shortages cost** = $\frac{\pi(t_1+t_2)w}{2}$
- **Holding cost** = $h\left[\frac{t_2(y+Z)}{2} + \frac{t_1-t_2)(Z+Z_1+py)}{2} + \frac{(t-t_1-t_3)Z_1}{2}\right]$

By utilizing the results of (2), (3) and (4), we obtain
TC = (c + d + Cs\theta p)y + k + \frac{h}{2}\left[ \frac{2 - D/\chi}{x} + \frac{(1 - p - D/\chi)^2}{D} \right]y^2

- \frac{h(1 - p)wy}{D} + \frac{(h + \pi)(1 - p)w^2}{2D(1 - p - D/\chi)} \quad (6)

Thus the expected total cost per unit time is given as

\[ E(TCP) = \frac{E(TC)}{E(t)} \]

\[ = \frac{D[c + d + CsE(\theta)E(p)]}{E_1} + \frac{kD}{E_1y} + \frac{hE_4y}{2E_1} - hw \]

\[ + \frac{(h + \pi)E_3w^2}{2E_1y} \]

where \( E_1 = 1 - E(p), E_2 = E\left( \frac{1 - p}{1 - p - D/\chi} \right), E_3 = E\left[ (1 - p - D/\chi)^2 \right]\) and

\[ E_4 = \frac{D(2 - \chi/\chi)}{x} + E_3 \]

Analysis and Numerical Example

The purpose of this paper is to solve the optimal values \( y \) and \( w \) to minimize the \( E(TCP) \), so that the optimal expected cycle length is accordingly obtained. To this end, we take the first and second-order partial derivatives of \( E(TCP) \) with respect to \( y \) and \( w \).

\[ \frac{\partial E(TCP)}{\partial y} = - \frac{2kD + (h + \pi)E_2w^2}{2E_1y^2} + \frac{hE_4}{2E_1} \quad (8) \]

\[ \frac{\partial E(TCP)}{\partial w} = -h + \frac{(h + \pi)E_2w}{E_1y} \quad (9) \]

\[ \frac{\partial^2 E(TCP)}{\partial y^2} = \frac{2kD + (h + \pi)E_2w^2}{E_1y^3} \quad (10) \]

\[ \frac{\partial^2 E(TCP)}{\partial w^2} = \frac{(h + \pi)E_2}{E_1y} \quad (11) \]

\[ \frac{\partial^2 E(TCP)}{\partial y \partial w} = - \frac{(h + \pi)E_2w}{E_1y^2} \quad (12) \]

Theorem 1. \( E(TCP) \) is strictly convex in \( y \) and \( w \).

Proof. From (10)–(12), the Hessian matrix of \( E(TCP) \) is given as
\[ H = \begin{vmatrix} 2kD + (h + \pi)E_2w^2 & -(h + \pi)E_2w \\ E_1y^3 & E_1y^2 \\ -(h + \pi)E_2w & (h + \pi)E_2 \\ E_1y^2 & E_1y \end{vmatrix} = \frac{2kD(h + \pi)E_2}{E_1y^4} \]

Hence we have \( H > 0 \) and \( \frac{\partial^2 E(TCP)}{\partial y^2} > 0 \) for all \( y \) and \( w \), and this completes the proof.

**Theorem 2.** The optimal values to \( E(TCP) \) are uniquely determined by

\[
y^* = \sqrt{2kD} \sqrt{h \left[ E_4 - \frac{hE_1^2}{(h + \pi)E_2} \right]}
\]

\[
w^* = \frac{hE_1y^*}{(h + \pi)E_2}
\]

**Proof.** This outcome is easily acquired by solving the first-order necessary conditions \( \frac{\partial E(TCP)}{\partial y} = 0 \) and \( \frac{\partial E(TCP)}{\partial w} = 0 \) simultaneously.

**Special Cases**

For the case of a constant \( \theta \), we have

\[
E(TCP) = \frac{D\left[ c + d + Cs\theta E(p) \right]}{E_1} + \frac{kD}{E_1y} + \frac{hE_4y}{2E_1} - hw + \frac{(h + \pi)E_3w^2}{2E_1y}
\]

and the same \( y^* \) and \( w^* \) as given in (13) and (14).

For the case that shortages are not allowed, we let \( \pi \to \infty \), then

\[
E(TCP) = \frac{D\left[ c + d + CsE(\theta)E(p) \right]}{E_1} + \frac{kD}{E_1y} + \frac{hE_4y}{2E_1}
\]

and \( y^* = \sqrt{\frac{2kD}{hE_4}} \), \( w^* = 0 \)

For the case that the ordered lot is 100\% perfect in quality, then we have \( p = 0 \), \( d \to 0 \) and \( x \to \infty \), such that \( E(p)=0 \) implying \( E_1 = E_2 = E_3 = E_4 = 1 \), and

\[
TCP = DC + \frac{kD}{y} + \frac{hy}{2} - hw + \frac{(h + \pi)w^2}{2y}
\]

and \( y^* = \sqrt{\frac{2kD(h + \pi)}{h\pi}} \), \( w^* = \sqrt{\frac{2khD}{(h + \pi)\pi}} \)

For the case that neither shortages nor defectives are allowed, then we have
\[ TCP = DC + \frac{kD}{y} + \frac{hy}{2} \]  

(20)

and \( y^* = \sqrt{\frac{2kD}{h}}, w^* = 0 \)  

(21)

**Example.** The following parameter values are partly adopted from those in Eroglu and Ozdemir’s (2007) paper. \( D = 15000, x = 60000, k = 400, h = 4, \pi = 6, c = 35, d = 1, Cs = 2. \) The rates of the defectives and the scraps are assumed to be uniformly distributed with the following probability density functions, respectively.

\[
f(p) = \begin{cases} 
10, & 0 \leq p \leq 1 \\
0, & \text{otherwise} 
\end{cases}
\]

\[
g(\theta) = \begin{cases} 
20, & 0 \leq \theta \leq 1 \\
0, & \text{otherwise} 
\end{cases}
\]

Consequently, \( E(p) = 0.05, E(\theta) = 0.10, E_1 = 0.95, E_2 = 1.36, E_3 = 0.49 \) and \( E_4 = 0.93 \), and from the Theorem 2, \( y^* = 2128.06 \) and \( w^* = 595.59 \). Also, from (1) and (7), we have optimal expected cycle length \( E(t)^* = 0.1348 \) in years and \( E(TCP)^* = 574,524.11/\text{year} \) respectively.

**CONCLUSION**

This paper tackled the inventory management problem in response to defective items with stock-out allowance. To meet all demands, an ordered lot undergoes a 100% screening measure to ensure a perfect quality before selling. The defective and the scrap rates are assumed to be randomly distributed, and furthermore, the objective to maximize the expected total profit per unit time is replaced with an equivalent one by minimizing the expected total cost per unit time, which is helpful for analysis. The close forms of \( y^* \) and \( w^* \) are derived, and we include many previous models as special cases. Regarding the shortages, the partial backlogging is a direction worth exploring more deeply for future research.

**REFERENCES**


