Pricing and Hedging of Currency Futures Options via the MEMM\textsuperscript{1}

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ABSTRACT

This paper validates the efficiency of both pricing and hedging for currency futures options based on the MEMM (Minimal Entropy Martingale Measure). It is well known that in the complete market, prices of derivatives are determined uniquely. However, since various constraints on trading exist in the real market, it is natural to see that the market is incomplete. So we use the MEMM both to price options and to construct hedge portfolios for the options under the mean-variance hedge. We can conclude that both pricing and hedging based on the MEMM have high performance, comparing that of the Garman-Kohlhagen model.

INTRODUCTION

According to (Harrison, 1981), in complete markets asset prices can be priced uniquely by an equivalent martingale measure. However, if the market is incomplete, it is impossible to deduce martingale measure uniquely. In the literature, some optimal martingale measures have been suggested. The Minimal Entropy Martingale Measure (MEMM) is one of them (see, e.g., (Frittelli, 2000)). The MEMM is a martingale measure that has the smallest entropy relative to the original objective probability measure. Therefore, the MEMM is not only good since it is the closest measure to the original measure under which prices of underlying assets evolve, but the MEMM is also consistent with utility maximization under the assumption that investors have exponential utilities (see, e.g., (Frittelli, 2000)). There are other martingale measures to price assets uniquely in incomplete markets. The measures include the Minimal Martingale Measure (MMM) (see, e.g., (Follmer, 1991)), and the Variance-Optimal Martingale Measure (v-MM) (see, e.g., (Schweizer, 1996,1997)). They are all the closest measures in the view of different metrics, and they are also consistent with expected utility maximization under specific forms of utility functions. In these martingale measures, for example, (Miyahara, 1999) tried to price options by the MEMM. In his paper, the price process of the underlying assets is assumed to follow a geometric Levy process.

In this paper we evaluate call options on currency futures using the MEMM. We empirically test efficiency of not only our pricing formula, but also mean-variance hedging for options based on the MEMM. However, without making assumptions on stochastic processes of the underlying assets, we nonparametrically estimate distributions of the underlying assets using the Kernel method which is a representative method of nonparametrical estimation.

The present paper is organized as follows. In section 2, we consider a method of change of the original objective probability measure to the MEMM. In Section 3, we derive option prices based on the estimated transition probability. In Section 4, based on the derived option prices in Section 3, we discuss the efficiency of risk-hedge methods for options. Finally, in Section 5 we empirically test efficiency of both our option pricing formula and performance of risk-hedge methods based on our pricing formula.

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We assume that uncertainty of the economy is represented by a given filtered probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). If the market is both frictionless and perfectly competitive, and there exists no arbitrage, then time- \( t \) discounted\(^{2}\) price \( \Pi_{t}^{\prime} \) of a European call option with maturity \( T \) and strike price \( K \) on the currency futures is given by

\[
\Pi_{t}^{\prime} = E^{\mathbb{Q}}\left[F_{t}(T) - K\right]\bigg|_{\mathcal{F}_{t}}
\]  

(1)

where \( F_{t}(t) := \text{time-} t \) price of the currency futures whose delivery time is \( \tilde{t} \), \( x^{\prime} := \max\{x, 0\} \), \( E^{\mathbb{Q}}\left[\mathcal{Y}_{t}\right] = \mathcal{Y}_{t} \) conditional expectation under the risk-neutral measure \( \mathbb{Q} \). Hereafter, for simplicity, we set \( \tilde{t} = T \) and denote \( X \) as \( F_{t}(t) \), and assume that \( X_{T} \) is a continuous random variable. Then pricing of call options is conditional to deriving the risk-neutral probability density function (p.d.f.) of \( X_{T} \). Through the paper, we also assume that the stochastic process \( \{X_{t}; t \in [0,T]\} \) has a Markov property and that the domestic interest rate \( r \) is constant. Then, under the risk-neutral probability \( \mathbb{Q} \), the stochastic process \( \{X_{t}; t \in [0,T]\} \) satisfies the following condition:

\[
E^{\mathbb{Q}}\left[e^{rT}X_{T}|X_{t} = x_{t}\right] = e^{rT}x_{t}
\]  

(2)

We denote \( p_{b}(x_{t}|x_{t}) \) and \( p_{b}(x_{t}|x_{t}) \) as the conditional p.d.f. under the risk-neutral probability measure \( \mathbb{Q} \) and the conditional p.d.f. under the original physical probability measure respectively. We can derive the conditional p.d.f. under the MEMM by solving such a mathematical programming as

\[
(PE) : \int p_{b}(x_{t}|x_{t})\log\frac{p(x_{t}|x_{t})}{p_{b}(x_{t}|x_{t})}dx_{t}, \quad \text{s.t.} \int e^{rT}x_{t}p(x_{t}|x_{t})dx_{t} = e^{rT}x_{t}
\]

Hereafter we simply refer to the conditional p.d.f. of \( X_{T} \) under the MEMM as the MEMM. That is, the MEMM is the solution for (PE). We can deduce next proposition by simple applications of Theorem 1 and p.270 of (Ben-tal, 1985).

**Proposition 1**

1. The MEMM is given by

\[
p^{\star}(x_{t}|x_{t}) = \frac{p_{b}(x_{t}|x_{t})e^{x_{t}}}{\int p_{b}(x_{t}|x_{t})e^{x_{t}}dx_{t}}
\]  

(3)

Here, \( \lambda \) is a Lagrangian multiplier corresponding to the constrained condition of Problem (PE).

2. The Lagrangian multiplier \( \lambda \) is given by

\[
e^{-rT}x_{t} = \frac{\frac{\partial}{\partial \lambda}E[e^{x_{t}}|X_{t} = x_{t}]}{E[e^{x_{t}}|X_{t} = x_{t}]}\]

(4)

where \( \tau = T - t \).

We know from Proposition 1 that for the derivation of the MEMM we need to estimate the p.d.f. \( p_{b} \) under the original probability measure. There are various methods for the estimate of \( p_{b} \), but we adopt the Kernel method since it is a representative method of nonparametric estimations. For simplicity, we use a Gaussian Kernel \( K(t) = \frac{1}{\sqrt{2\pi}}e^{-t^{2}/2} \).

\[\text{We assume that for any } u \in \mathbb{R}, \{X_{u+T-t}, X_{v}\} \text{ has an identical probability distribution. This assumption makes it possible to estimate } p_{b}(x_{t}|x_{t}) \text{ by the Kernel method using a dataset } \{(X_{u+T-t}, X_{v}); i = 1 \cdots n\}. \]

Using the Kernel method, the estimated conditional p.d.f. \( p_{b}(x_{t}|x_{t}) \) is given by

\[\text{Here and hereafter we refer to discounted as discounted by the numeraire price. And we set numeraire price at time } t \text{ be } e^{rT}, \text{ where } r \text{ denotes the domestic interest rate that is assumed to be constant. Hereafter we refer to option prices, prices of currency futures, strike prices, and so on, all mean discounted prices.} \]
\[
\hat{p}_b(x_t | x_i) = \frac{1}{n h^2} \sum_{i=1}^n K \left( \frac{x_t - X^i_t}{h} \right) K \left( \frac{x_i - X^i_t}{h} \right)
\]
\[
\hat{p}_b(x_t | x_i) = \frac{1}{n h} \sum_{i=1}^n K \left( \frac{x_i - X^i_t}{h} \right)
\]

where \( h \) is the optimal bandwidth, that is \( h = n^{-\frac{1}{2+4}} \) (see, e.g., (Silverman 1996) section 4.3.2).

**Lemma 1.2**

Using Gaussian Kernel, (assume \( p_b \) is estimated by the Kernel method) the Lagrangian multiplier \( \lambda \) is given by,

\[
e^{x^T_x} = \frac{1}{n h} \sum_{i=1}^n K \left( \frac{x_i - X^i_t}{h} \right)
\]

\[
e^{x^T_x} = \frac{1}{n h} \sum_{i=1}^n e^{x^T_x} K \left( \frac{x_i - X^i_t}{h} \right)
\]

**Proof of Lemma 1.2**

If we estimate \( p_b \) by \( \hat{p}_b \) through (5) and calculate expectation of the right hand side of (4),

\[
E[e^{x^T_x} | X_t = x_t] = \frac{1}{n h} \sum_{i=1}^n e^{x^T_x} K \left( \frac{x_i - X^i_t}{h} \right)
\]

\[
\frac{\partial E[e^{x^T_x} | X_t = x_t]}{\partial \lambda} = \frac{1}{n h} \sum_{i=1}^n e^{x^T_x} K \left( \frac{x_i - X^i_t}{h} \right)
\]

This leads to the result. Q.E.D.

**Remark 1.3**

We derive \( \hat{p}_b \) as an estimate of \( p_b \) through the Kernel method. Note that using another p.d.f. leads to a different result. Since we want to test efficiency of the MEMM as fairly as possible, we do not assume a specific stochastic model but adopt nonparametric estimation.

**Proposition 1.4**

An estimated price \( \overline{P}_t \) of the call option on the currency futures is given by

\[
\overline{P}_t = \frac{\sum_i h e^{x^T_x} \left( A_i \Phi(A_i) + \phi(A_i) \right) \left( \frac{x_i - X^i_t}{h} \right) / \left( \sum_{i=1}^n e^{x^T_x} \phi \left( \frac{x_i - X^i_t}{h} \right) \right)}
\]

where \( A_i := \lambda h + \frac{X^i_t - K}{h} \), \( \Phi \) := standard normal distribution function, and \( \phi \) := standard normal p.d.f.

**Proof of Proposition 1.4**

If we set an estimate of p.d.f. \( p_b(x_t | x_i) \) as \( \hat{p}_b(x_t | x_i) \), then

\[
E^0(X_t - K)^+ | X_t = x_t = \frac{\int (x_t - K) e^{x^T_x} p_b(x_t | x_i) dx_t}{\int e^{x^T_x} p_b(x_t | x_i) dx_t}
\]

\[
= \frac{1}{n h} \sum_{i=1}^n \int (x_t - K) e^{x^T_x} \phi \left( \frac{x_t - X^i_t}{h} \right) \phi \left( \frac{x_i - X^i_t}{h} \right) dx_t
\]

\[
= \frac{1}{n h} \sum_{i=1}^n \int e^{x^T_x} \phi \left( \frac{x_t - X^i_t}{h} \right) \phi \left( \frac{x_i - X^i_t}{h} \right) dx_t
\]

Note that we replace \( p_b \) to p.d.f. \( \hat{p}_b \) that is estimated by the Kernel method on the second equality of the above equation. Furthermore, since
\[ \int \frac{\phi \left( \frac{x_t - X_{t-1}}{h} \right)}{\Phi \left( \lambda h + \frac{X_{t-1} - K}{h} \right)} dX_t = \int \frac{\phi \left( \frac{x_t - X_{t-1}}{h} \right)}{\Phi \left( \lambda h + \frac{X_{t-1} - K}{h} \right)} d_x = \int \phi \left( \frac{x_t - X_{t-1}}{h} \right) d_x = \int \phi \left( \frac{x_t - X_{t-1}}{h} \right) d_x = \int \frac{1}{\Phi \left( \lambda h + \frac{X_{t-1} - K}{h} \right)} dX_t \]

This holds, the proof is completed. Q.E.D.

### THE MEAN-VARIANCE HEDGE

Unless the market is complete, making a perfect risk-hedge is tricky. Instead, in an incomplete market, there are some methods for the risk-hedge through various risk measures. In this paper, we consider a mean-variance hedging (hereafter, MVH). We assume that investors reconstruct their portfolio at discrete times \( k = 0,1,\cdots,T \); then the MVH is given as a solution of the following minimizing problem:

\[ \min_{c,\theta_G(\theta)} E \left[ (H - c - G_1(\theta))^2 \right] \]  

Here we set \( G_i(\theta) := \sum_{j=1}^{i} \theta_j \Delta X_j \), \( t = 0,1,\cdots,T \). \( \Delta X_j := X_j - X_{j-1} \). An analytic solution of (8) is generally possible only if the mean-variance trade-off process (MVT) is bounded and deterministic. Here, the MVT is defined as \( \sum_{j=1}^{t} \left( \frac{E[\Delta X_j|X_{j-1}]}{\text{Var}_{\Delta X_j}[X_{j-1}]} \right) \). It is known that if the MVT is deterministic, then the MMM, the v-MM, and the MEMM are all identical (see, e.g., (Schweizer 1995b)). It is also known that the most consistent martingale measure of the mean-variance hedge is the v-MM. Thus, we assume that the MVT is bounded and deterministic. According to (Schweizer 1995), if the MVT is deterministic, then a solution of (8) is given by

\[ \theta_t = \xi_t + \alpha_t \left( \hat{V}_t - c - G_{i-1}(\theta) \right) \]  

\[ c = V_0 \]  

(10)

where

\[ \xi_t := \frac{\text{Cov}_{\pi_t} \left( \Pi_t - \sum_{j=1}^{T} \xi_j \Delta X_j, \Delta X_t \right)}{\text{Var}_{\pi_t} (\Delta X_t)} \]

(11)

and where \( \hat{V}_t = \text{E}[(H - c - G_{i-1}(\theta))] \).

At first appearance, it is tedious to calculate \( \xi_t, \alpha_t \). However, since the following equations

\[ \text{Var}_{\pi_t} (\Delta X_t) = \text{Var}_{\pi_t} (\Delta X_t) = \text{Cov}_{\pi_t} (\bullet, \Delta X_t) = \text{Cov}_{\pi_t} (\bullet, X_t) \]

hold, we can deduce

\[ E_{i-1} \left[ \left( \Pi_t - \sum_{j=1}^{T} \xi_j \Delta X_j - E_{i-1} \left( \Pi_t - \sum_{j=1}^{T} \xi_j \Delta X_j \right) \right) (X_t - E_{i-1} \left( X_t \right)) \right] \]

(11)

Furthermore, estimating \( p_h \) by the Kernel method with the Gaussian Kernel, we obtain, for \( t > s \)

\[ E_t [X_s] = \int x \phi \left( \frac{x_t - X_t}{h} \right) dx_t = \int \phi \left( \frac{x_t - X_t}{h} \right) dx_t = \sum_{j=1}^{T} \phi \left( \frac{x_t - X_t}{h} \right) dx_t \]

(12)

Applying (10) to (9), the calculation of \( \xi_t \) becomes straightforward. Similarly, we can calculate \( \alpha_t \) by using (10). We note that the most R.H.S. of (10) is called the Nadaraya-Watson estimator which is used in nonparametric regressions.
EMPIRICAL RESULTS

Based on the results derived in the previous sections, we test our models empirically. We need time series data of currency futures and the domestic interest rate to estimate option prices under the MEMM from Proposition 1. We assume the interest rate to be constant. We use the 3 month Yen repo rate as the interest rate for the empirical tests. Time series covers the time period from 2004/2/27 through 2004/8/11, indicating sufficiently large sample size of 121.

To estimate the Yen/Dollar currency futures options we use data of currency futures with the same maturity date during the above sample period. For example, to estimate the price of the option with maturity 2004/Sep., we use data of the currency futures whose delivery month is 2004/Sep. Table 1 lists basic statistics of the sample data of the currency futures. We take a sample of time series data from Thomson-Financial-Data-Stream.

Table 1: Basic Statistics

Let us plot a p.d.f.s estimated from these sample data. For an example, sample data of the currency futures with delivery month 2004/Sep. and \( t = 2004/7/28 \) is used. Since \( p_\beta(x_t|x_t) \) is bi-variate and the figure is three-dimensional, it is hard to discern its features. Therefore, after estimating the conditional p.d.f., we take a cross-section of the figure at some value of \( X_t \). Since the price of currency futures with delivery month 2004/Sep. at \( t = 2004/7/28 \) is 89.68, we fix \( X_t \) to be 90.00 (see Figure 1). We estimate the p.d.f. via 60-day sample data before 2004/7/28. A histogram and the estimated p.d.f. are depicted in Figure 1. Since the p.d.f. estimated by the Kernel method is generated from the histogram, it can be confirmed whether the p.d.f. is well fitted to the histogram. It can be seen that the estimated p.d.f. relatively fits the histogram. Further, the non-negativeness of the underlying asset can be confirmed from Figure 1.

**Figure 1: The estimated p.d.f. \( \hat{p}_\beta(x_t|x_t) \) at \( t = 2004/7/28 \) and \( T = 2004/9/3 \)**

Performance of the MEMM Option Price

We consider an estimated price of the call option whose maturity \( T \) is 2004/Sep and the strike price \( K \) is 90. Figure 2 illustrates the estimated option prices under the MEMM as well as the estimated prices of the Garman and
Kohlhagen model (GK).

Figure 2: Strike Price 90, Maturity 2004/Sep.

Under the GK model option price \( \Pi' \) is given by
\[
\hat{\Pi}' = X_t \Phi(d) - K \Phi(d - \sigma \sqrt{\tau})
\]
where \( d := \left[ \ln(X_t/K) + \left( \sigma^2/2 \right) \tau \right] / \sigma \sqrt{\tau} \) and \( \sigma \) is the volatility of the price process \( X \) of the underlying asset.

Here we note that \( X, K, \Pi' \) are all discounted values. We then discuss how the volatility should be given. Two approaches are available. The one uses the historical volatility which is a standard deviation of sample data of the price process of the underlying asset. The other is the implied volatility that is a standard deviation under which the GK option price coincides to that of the market. If we use the implied volatility under the GK model, there would be no difference between the market price and the estimated price other than the difference generated from the difference between the maturity of the call option and the delivery month. We make use of not only the GK prices with the implied volatility, but also the ones with the historical volatility to compare the performance of those prices with that of the MEMM prices. We calculate the historical volatility from the same sample data used to estimate the option prices under the MEMM. Since the prices under the MEMM are estimated from only the given sample data of currency futures, it is fair to compare the estimate prices of the GK model with the historical volatility against the estimated prices under the MEMM.

We estimate the option price during the period from 2004/6/17 to 2004/8/11. In this period trading volumes are only at most 10 until 2004/7/7. After 2004/7/8, the trading volumes gradually increase. On 2004/7/28 and 2004/8/3, the trading volume is 442 and 400, respectively, with the highest volume being recorded around these dates. Figure 2 shows that the prices of the GK with the historical volatility have low performance compared with the ones under the MEMM or the ones of the GK with the implied volatility, either of which is near the market prices of the option. The performance of the prices of the GK with the historical volatility is low, in the case of high or low strike price, and in the case of long or short maturity. This may show that the prices under the MEMM are better than the prices of the GK from the point of view models comparison. It is difficult to determine the difference of performance between the prices of the GK with the implied volatility and the prices under the MEMM only through Figure 2. Thus, Figure 3 is the same figure as Figure 2, except that the prices of the GK with the historical volatility are removed.

\( ^3 \) The GK is a version of Black-Scholes model specified to currency options and a very popular pricing model among practitioners. See (Garman, 1983).
Figure 3: Strike Price 90, Maturity 2004/Sep.

From Figure 3, we can see that both the prices under the MEMM and the ones of the GK model with the implied volatility are near the market prices. A tendency that the longer to the maturity, the nearer the prices under the MEMM are to the market prices. Since the options that we estimate the prices are ATM ("At the Money") that have less explanatory power by the intrinsic value especially in a period far from the maturity, the prices under the MEMM have more explanatory power except for the intrinsic values. This may show the advantageousness of the MEMM. Although we test their performances under more various maturities, such as 2004/Sep., 2004/Dec., 2005/Mar., etc., obtained results are not reported here due to similarity of results and space limitations. Results show that the lower the strike price is, the nearer the performance of prices under the MEMM is to that of the GK model with the implied volatility. Nevertheless the intrinsic value has high explanatory power in these cases, which lessen the difference between the prices under the MEMM and the one of the GK model. In so far as in these cases the prices of the GK model with historical volatility have lower performance, it can be said that the prices under the MEMM appear superior to those of the GK model.

PERFORMANCE OF THE MEMM ON RISK-HEDGE

Various measures for evaluating performance of a hedge portfolio are available. In this paper, we consider performance of the hedge portfolio in the view of how accurately the value of the hedge portfolio replicates the option price. We construct hedge portfolios through the estimated option prices under the MEMM. We consider a mean-variance hedge which is one of hedge methods in incomplete markets. It is known that the most consistent martingale measure with the mean-variance hedge is the v-MM (see, e.g., (Schweizer, 1995)), however since we assume that the process of the mean variance trade-off (MVT) is deterministic, the MEMM coincides with the v-MM (see, e.g., (Schweizer, 1995b)). To test the performance of the mean-variance hedge, we construct a hedge portfolio under the delta hedge through the prices of the GK model. Next we compare the performance of the hedge portfolios under the mean-variance hedge, with that of the hedge portfolio under the delta hedge. We test hedge portfolios under both hedge methods against the call option with maturity 2004/Sep and strike price 90 on the currency futures. Column (3), (4) and (5) of Table 2 report value of the hedge portfolio. Since both mean-variance hedge and delta hedge are self-financing, portfolio value $V_t$ at time $t$ is given by $V_t = c + \sum_{i=1}^{\theta} \Delta X_i$. In a case of mean-variance hedge, $c$ and $\theta$ are given by the equation (9). On the other hand in a case of delta hedge $c$ is the option price at the initial date and $\theta$ is the option's delta. In the empirical test we set the initial date on which each hedge portfolio is set up 2004/8/3. The bold characters on Table 2 show that the performance of the mean-variance hedge is the best (see, 2004/8/4, 2004/8/5, 2004/8/6, and 2004/8/9 on Table 2.). That is, the difference between the portfolio value under the mean-variance hedge and the market price of option is smaller than any other hedging portfolio. In the period, prices of currency futures are about between 90 and 91, therefore, it can be said that the options are ATM.
Table2: Hedge-performance for the option with strike price 90 and maturity 2004/Sep

Although we test the hedge portfolio against the option with same maturity and strike price 91, the result is similar to the case of strike price 90. That is, the hedge portfolio under the mean-variance hedge shows the best performance. As for the cases of other maturities and other strike prices, if the options are ATM or OTM, then the mean-variance hedge shows high performance.

CONCLUSIONAL REMARKS

From the above discussion, we can see that option prices estimated under the MEMM have not only higher performances than the option prices estimated by the GK with historical volatilities, but also high performances as the option prices estimated by the GK with the implied volatilities. It is remarkable that the MEMM makes as good performances as the GK with the implied volatility; since pricing errors of the GK are caused only by the difference between the option maturity and the delivery month of the currency futures.

It is also shown that the mean-variance hedge under the MEMM is better than the delta hedge under the GK. This tendency is especially obvious in the case of high strike prices. Given the limited computational ability of available PC, we could test only the cases of 6-days to maturity. However, improving computational ability will enable us to test cases of longer time to maturity. In the near future, we will try to develop more sophisticated pricing and hedging methods under the MEMM, and we will test performances of the MEMM using more various contingent claims.

REFERENCES


