ABSTRACT

The purpose of this paper is to explore the similarity of Nash equilibrium between Cournot and Bertrand duopoly game. The aim is separately to investigate the equilibrium of output competition and price competition and then to find out whether there is any coincident equilibrium between the output game and the price game. From the literature review the basic outline of two games are focused and necessary equations are derived. With the computer and its software, figures are plotted and the equilibriums are figured out by logic reasoning. The results showed Bertrand game is more complex than Cournot game, while a coincident Nash equilibrium exist if the output constrain is set in the Bertrand game. The results of the study provide evidences that Nash equilibriums exist in different games, equations that are derived in the study can be used for further research, and suggest that the simulated game can be processed by computer.

Keywords: Nash equilibrium, Cournot game, Bertrand game, repeated competition, mixed strategy.

INTRODUCTION

Game theory is concerned with how individuals make decisions when they are aware that their actions affect each other and when each individual decision makers, all of whom are behaving purposefully, and whose decisions have implications for other people that makes strategic decisions different from other decisions (Bierman and Fernandez, 1998; Rasmusen, 1995). In general, the game has the assumptions: (1) there are at least two firms or players producing homogeneous products; (2) firms do not cooperate; (3) firms have the same constant marginal cost; (4) demand is linear; (5) firms compete in output, price, or the other variables, and choose their respective outputs or prices simultaneously; (6) there is strategic behavior by both firms; (7) both firms compete solely at output or price; (8) consumers buy every thing from the cheaper firm or half at each, if the price is equal.

The essential elements of a game are players, strategies, payoffs, actions, information set, outcomes, and equilibriums. At a minimum, the game description must include the first three items: players, strategies, and payoffs (Friedman, 1991; Rasmusen, 1995). In game theory, one of the fundamental equilibrium is Nash equilibrium. At Nash equilibrium, each player’s equilibrium strategy is a best response to the belief that the other players will adopt their Nash equilibrium strategies (Bierman and Fernandez, 1998; Teng and Thompson, 1983; Turocy and von Stengel, 2001). That means if a player stands at a price or output position, the other player will not get his correspondently best payoff when he chooses different strategy. Thus, to find a game solution or equilibrium is not a maximum problem, but a peculiar and disconcerting mixture of several conflicting sub-maximum problems (von Neumann and Morgenstern, 2004).

Cournot wrote the book “Researches into the Mathematical Principles of the Theory of Wealth” to discuss the competition of how firms would choose their output in an attempt to maximize profit (Garth, 1991; Walker, 1995). Conventionally, it is to call these situations “games” when they are being studied from an abstract mathematical viewpoint, and hence the original situation is reduced to a mathematical description or model (Davis, et al. 2003; Nash, 1953; Saloner, 1991). Although Cournot model precedes the invention of game theory, it is an important predecessor to the later work of game theory forerunners (Bierman and Fernandez, 1998). In fact, Cournot model forms a mathematical theory that deals with the situation that the final outcome depends primarily on the combinations of strategies selected by the adversaries in a formal reasonable way, which we may say that it is a kind of game.

Many researchers studied Cournot game with Nash equilibrium in various fields with different topics and the findings are abundant (Branzei, et al. 2003; Chen, 2007a; Davis, et al. 2003; Tasnadi, 2006; Thille, 2003).
Cournot-Nash equilibrium model, price is not a strategic variable that is the market price determined by the total output produced by all the firms in an industry, no individual firm directly controls exactly the same market price (Bierman and Fernandez, 1998; Binmore, 1992; Friedman, 1991; Rasmusen, 1989). Cournot game assumes that the quantity each firm produces affects the market price (Davis, et al. 2003); while in the real world, firms usually choose their own price and the consumers determine the firms’ sales (Friedman, 1991; Kreps, 1995; Walker, 2005). Bertrand game is a model of price competition between duopoly firms that results in charging the price separately that would be charged under perfect competition, known as marginal cost pricing \( c \) (Beckman, 2003; Binmore, 1992; Chen, 2007b; Rasmusen, 1995). Competing in price means that firms can easily change the quantity they supply, but once they have chosen a certain price, it is very hard to change it (Bierman and Fernandez, 1998). The Bertrand solution is Nash equilibrium in price while the Cournot solution is Nash equilibrium in output.

Bertrand price competition is a kind of discontinuous game. Levitan and Shubik (1972) introduced two firms compete with price as the strategic variable and in which the firms are limited by capacity constraints. After that, the theory of existence of equilibrium in discontinuous economic game is presented (Dasgupta and Maskin, 1986; Reny, 1999). Few papers present Bertrand game from the zero-profit pricing strategy \((c^*, c^*)\) to the capacity constraint pure strategy then the capacity constraint mixed strategy. This paper uses the computer thoroughly to investigate pure and mixed solutions of Bertrand games, and compares these solutions to Cournot output game to find whether any similarity or coincident equilibrium between two games. What follows in the passage is Cournot game which is the simpler one of the two games.

**COURNOT GAME**

The Cournot model has the following features: (a) there is more than one firm and all firms produce a homogeneous product; (b) firms do not cooperate; (c) firms have market power; (d) there are barriers to entry; (e) there is strategic behavior by the firms; (f) firms compete in quantities, and choose quantities simultaneously. The essential assumption of Cournot model is that each intelligent and rational firm aims to maximize profits, based on the expectation that its own output decision will not have an effect on the decision of its rivals at a time (Chen, 2007a). All firms know the total number of firms in the market, and take the output of the others as given. Each firm takes the quantity set by its competitors as a given, evaluates its residual demand, and then behaves as monopoly. The cost \( c \) is common knowledge (Athey and Bagwell, 2001). In game theory, something is common knowledge if everybody knows it; everybody knows that everybody knows it; everybody knows that everybody knows that everybody knows it; and so on. All outputs appear on the market simultaneously. The price for the product always equals the market clearing price, and the market price per unit of output, \( p \), received by all firms is a decreasing function of the total output, \( q \), produced by all firms. If a market has \( n \) producers, these producers play a simultaneous move game in which each player chooses a number \( q_i \) from the interval \([0, M]\). If the firms sell their product, denoted \( q_i \) every day at an auction, the price set at the auction is given by:

\[
p = M - q = M - \left( \sum_{i=1}^{n} q_i \right).
\]

If \( M \) is a much larger number than the cost \( c \), the profit for the \( i \)'s firm is

\[
\pi_i(q_1, \ldots, q_i, \ldots, q_n) = (p - c)q_i = [M - (\sum_{i=1}^{n} q_i) - c]q_i.
\]

**Solution of Cournot Duopoly Game**

In Equation 2, if \( i = 1, 2 \), it is a duopoly game. A solution of the game means a determination of the amount of satisfaction each individual should expect to get from the situation and should not deviate from the rationally best situation (Nash, Jr., 1950). By partial derivative, solving for the firms best response functions, \( n \) equations will be get. The firm \( i \)'s equation is

\[
q_i + \sum_{i=1}^{n} q_i = M - c.
\]
Solving the \( n \) best response functions, the output for each firm is
\[
q_i^* = \frac{1}{n+1}(M - c). \tag{4}
\]
This output, the strategy profile \((q_1^*, \ldots, q_n^*)\), was defined as the Cournot-Nash equilibrium because this is the Nash equilibrium of a Cournot game (Bierman and Fernandez, 1998; Binmore, 1992; Rasmusen, 1995). At the equilibrium, each firm’s maximized profit and price is
\[
\pi_i^* = \frac{1}{(n+1)^2}(M - c)^2. \tag{5}
\]
\[
p_i^* = \frac{1}{n+1}(M + nc). \tag{6}
\]
Equation 4, 5, and 6 are an equilibrium solution set.

If \( n = 1 \), it is a monopoly solution. When \( n = 2 \), it is a two-player duopoly game. If \( n \) is equal or greater than 3, the game has the oligopoly result. If \( n \to \infty \), it is a perfectly competitive game; the total output is \( M - c \), the price is \( c \), and each play’s profit is \( 0 \).

**Duopoly Output Game**

For numerical example, when \( n = 2 \), it is a two players duopoly game. Letting \( M = 12.1 \), \( c = 0.1 \), \( M - c = 12 \); \( q_1 \) and \( q_2 \) is the output of firm-1 and firm-2; from Equation 2 we have
\[
q_2 = 12 - q_1 - \frac{\pi_1^*}{q_1}. \tag{7}
\]
Giving different values of \( \pi_1 \), the iso-profit curves for the firms are shown on Figure 1. Based on the Equation 3, at the equilibrium, the strategy profile is \((q_1^*, q_2^*) = (4, 4)\), and based on Equation 7, the profit for each firm is \( 16 \) and the price is \( 4.1 \).

![Figure 1: Duopoly iso-profit (pi) curves and reaction lines (R1 and R2).](image)

**Duopoly Repeated Game**

Suppose now that the duopoly player deviate from their equilibrium output or they are not fully rational at the first. The repeated response becomes a repeated game. Two players always proceed on the assumption that the other firm will produce this period exactly what the opponent produced last period. At the time \( t = 0 \), the firm-1 produces \( q_1^0 \)
and the firm-2 produces \( q_2^0 \). Later, if the firm-2 assumes that the firm-1 will also produce \( q_1^0 \) in period \( t = 1 \), the firm-2 has her best reply \( q_2^1 \), than the firm-1 has the response \( q_1^1 \), and so on. Supposing the initial outputs are the following:

\[
t = 0 \begin{cases} \quad q_1^t = a \\ \quad q_2^t = b. \end{cases} \tag{8}
\]

Setting \( N = \sum_{i=1}^{t} (-1)^i (0.5)^i \), when \( t \) equals odd and even, the response functions are

\[
\begin{cases} \quad q_2^t = N(M - c) - (0.5)^i a \\ \quad q_1^t = N(M - c) - (0.5)^i b \\ \quad q_2^t = N(M - c) + (0.5)^i b \\ \quad q_1^t = N(M - c) + (0.5)^i a. \end{cases} \tag{9}
\]

\[
\begin{cases} \quad q_2^t = N(M - c) - (0.5)^i a \\ \quad q_1^t = N(M - c) - (0.5)^i b \end{cases} \tag{10}
\]

If \( t \) approaches to infinite, \( N \) converges to one third and \( (0.5)^i \) approaches to zero. Because \( M - c \) is equal to 12, \( q_1^t \) and \( q_2^t \) are eventually equal to (4, 4), as the equilibrium strategy profile. By using computer numerical investigation, Figure 2 shows the initial points, path of convergence, after ten runs of competition, two firms reach at the equilibrium (4, 4).

![Figure 2: Four repeated game, \((q_1, q_2)\) converge to the equilibrium (4, 4).](image)

**Discussion of Duopoly Cournot Game**

If two profit-maximizing firms do not like competition game, they may collude to shun the output competition. They share the output \( q_1 = q_2 = 0.5(M + c) = 6 \) and set \( p = 0.5(M + c) = 6.1 \) then each firm gets the profit \( (p - c)q_1 = 36 \).

However, this is not a make-sure deal because any collusion deal is inherently unstable. Often the deals will be illegal. As Figure 2 displayed, after the repeated response, the only deal on which neither has an incentive to rat is the output pair (two third of \( q_1 \) and \( q_2 \)) that equals the equilibrium strategy profile (4, 4). Based on the definition of Nash equilibrium, for two-player game, the strategy profile (4, 4) is the only solution, because if each player has chosen a strategy and no player can benefit by changing his or her strategy while the other player keeps his or her unchanged.

**BERTRAND GAME**

This is a price competition game. Competing in price means that firms can easily change the quantity they supply, but once they have chosen a certain price, it is very hard to change it (Bierman & Fernandez, 1998). Suppose the firm-1

![Figure 2: Four repeated game, \((q_1, q_2)\) converge to the equilibrium (4, 4).](image)
and firm-2 announce prices without knowing what price the other firm will announce. The cost $c$ for either firm is common knowledge. The profits are based on $c$, the total output $q = q_1 + q_2$, and the demand function that is
\[ p(q) = M - q. \]  
(11)

In the Cournot game, firms choose outputs and the market price is not a variable; in the Bertrand game, they choose prices and sell as much as they can. The profit function for each firm is
\[ 
\pi_i = \begin{cases} 
(p_j - c) \cdot (M - p_i) & \text{if } p_i < p_j \\
\frac{1}{2} (p_j - c) \cdot (M - p_i) & \text{if } p_i = p_j \\
0 & \text{if } p_i > p_j.
\end{cases} 
\]  
(12)

Equation 12 means that if the manager of firm-1 thinks firm-2 choose a price that is above their cost, it is in his best interest to charge a price just slightly below the price of his rival. Then, the firm-1 will capture the entire market. Because price exceeds constant marginal cost, the firm-1 will earn additional profits on each of the goods sold. If the manager of firm-1 chooses a price higher than the price of his rival, then he will not sell any product and earn no profits. If the manager of firm-1 charges a price identical to his rival, he will only sell the drinks to half of the market. Because each firm knows the other firm will be best served by charging a price slightly below their rival’s, the non-cooperative Nash equilibrium price is $c$ (Bierman and Fernandez, 1998; Reny, 1999). With the definition of Nash equilibrium, each player has chosen a strategy and no player can benefit by changing his or her strategy while the other player keeps his or her unchanged (Chen, 2007b).

### Pure Strategy Solutions to Capacity Constraint

If the firms have constrained productive capacity, the simple Bertrand game becomes complicated. The solutions may differ in the capacity constraint and the number of firm (Peters, 1984). A critical capacity and price for pure equilibrium strategy can be obtained by investigating numerical data and figures based on the definition of Nash equilibrium. In this section a found critical capacity will divide the solution into pure and mixed strategy. These will be stated in the following.

Assuming both firms are constrained to charge the same price, the market’s demand for the drinks is
\[ p = \begin{cases} 
M - (q_1 + q_2) & \text{if } q_1 + q_2 < M \\
0 & \text{if } q_1 + q_2 \geq M.
\end{cases} 
\]  
(13)

If both firms can only supply $0 < L < M$ units, and this is common knowledge, the sales of the firms as a function of both prices is
\[ q_i(p_1, p_2) = \begin{cases} 
\min(M - p_1, L) & \text{if } p_1 < p_2 \\
\frac{1}{2}(M - p_1, L) & \text{if } p_1 = p_2 \\
\max(0, \min(M - p_1 - L, L)) & \text{if } p_1 > p_2 \\
0 & \text{otherwise.}
\end{cases} 
\]  
(14)

The equation means that, when the firm-2 sets its price above that of the firm-1, the firm-1’s sales are constrained by the lower of the market demand and the capacity constraint. If both firms choose the same price, sales are constrained by the lesser of half of the market demand at the price chosen or the capacity constraint. If firm-1 sets its price above that of firm-2, the firm-1’s sales will be constrained by the higher of zero output or the lesser of the residual demand and the capacity constraint (Bierman and Fernandez, 1998; Stahl II, 1988). Following Equation 14, the profit function of the firm-1 is
\[ \pi_1(p_1, p_2) = (p_1 - c) \cdot q_1(p_1, p_2) \]  
(15)

To explore numerically, let $M = 12.1$, $c = 0.1$. In Figure 3 (constrained output $L = 4.0$), the firm-2 charges $p = 4.1$, if the firm-1 charges 4.0 ($p_1 < p_2$) or charges 4.2 ($p_1 > p_2$), then the firm-1 will get less profits than 16. Therefore, the firm-1 will Charge $p = 4.1$. Similarly, when $L = 4.0$, the firm-1 charges $p = 4.1$, if the firm-2 charges 4.0 ($p_1 > p_2$) or charges 4.2 ($p_1 < p_2$), then the firm-2 will get less profits than 16. Therefore, the firm-2 will Charge $p = 4.1$. 

Mixed strategy solution for equal capacity constraint

If two firms capacities, $L$, are more than 4.0 that is one third of $(M - c)$, the sales of the firm-1 is

$$q_1(p_1, p_2) = \begin{cases} \min(M - p_1, L) & \text{if } p_1 < p_2 \\ \max(0, \min(M - p_1 - L, L)) & \text{if } p_1 > p_2. \end{cases}$$

(16)

Assuming $f_2(p)$ is the probability distribution function of the price of the firm-2 (Levitan and Shubik, 1972). The firm-1 expected sale is

$$E(q_1) = (1 - f_2(p_1)) \cdot \min(M - p_1, L) + f_2(p_1) \cdot \max(0, \min(M - L - p_1, L)).$$

(17)

The firm-1’s profit is

$$\pi_1 = (p_1 - c) \cdot E(q_1).$$

(18)

Because $M - L - p_1 > 0$, $M - p_1 > L$. From the demand function $p = M - q_1 - q_2$, and because $q_1 = q_2 = L$, than $L = M - L - p$, the price $p$ has to be $p_1 = p_2 = p$. Because $p_1 > p_2$ breaks the equal quantity, the result is $L > M - L - p_1$. For generalization, Equation 18 can be rewritten as

$$\pi_1(p) = (p - c)[(1 - f_2(p))L] + f_2(p)(M - L - p)].$$

(19)

Solving for $f_2(p)$, one obtains

$$f_2(p) = \frac{L(p - c) - \pi_1(p)}{(p - c)(p + 2L - M)}.$$  

(20)

The profit depends on the $M, L$, and $c$ as the monopoly that constrained by the other firm’s capacity $L$; it is

$$\pi_2(p) = \frac{(M - L - c)^2}{4}.$$  

(21)

Substitute Equation 21 into 20, than the probability function is

$$f_2(p) = \frac{4L(p - c) - (M - L - c)^2}{4(p - c)(p + 2L - M)}.$$  

(22)

Figure 3, using Equation 22 with $M = 12.1$ and $c = 0.1$, shows how the probability and price change for the different values of capacity constraint. For example, if constrained capacity is equal to 6.0, the probability for both firms to charge $p = 1.86$ is equal to 0.5. From Equation 19, Figure 5 (capacity constraint $L = 4$) displays if the firm-2 chooses different probability, the intersection is at price 4.1 and profit 16. The intersection is the same as the pure strategy shown in Figure 3. This means if the firm-1 charges price 4.1, the firm-2 for all possibilities will charge the same price.
Mixed Strategy Solution for Different Capacity Constraint

If the price game is unequal-capacity constraint, referring to Equation 24 and 25, any of the firm expected sale is

\[
E(q_i) = (1 - f_j(p)) \cdot \min(M - p, L_i) + f_j(p) \cdot \max[0, \min(M - L_j - p, L_i)].
\]  

(23)

The profit for firm-1 or 2 is

\[
\pi_i = (p - c) E(q_i).
\]  

(24)

Specifically, the profits depend on the \( M, L \), and \( c \) as the monopoly that constrained by the other firm’s capacity \( L \); it is

\[
\pi_i = \frac{(M - L_j - c)^2}{4}.
\]  

(25)

The probability function may be written as

\[
f_j(p) = \frac{4(p - c) \cdot \min(M - p, L_i) - (M - L_j - c)^2}{4(p - c)K},
\]  

where \( K = \{\min(M - p, L_i) - \max[0, \min(M - L_j - p, L_i)]\} \).

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Figure 4: The probability and price change for the different values of capacity constraint \( L \).

Figure 5: The firm-2 chooses different probabilities. The intersection is at price 4.1 and point 16.
From Equation 26, the mixed strategy \([f(p), f(p)]\), can be numerically solved and plotted by letting \(M = 12.1, c = 0.1\), price, and constrained capacities. Figure 6 displays the distribution of probability versus price. In Figure 6, the pair \((9, 7)f_1, (9, 7)f_2\) means firm-1 and firm-2 probabilities, if the capacity constraint for firm-1 is 9 and for firm-2 is 7. The resemble meaning is for the pair \((6, 5)f_1, (6, 5)f_2\). For example, the mix strategies are \([0.07 f_1(1.0) = 0.92, 0.07 f_1(1.0) = 0.42]\) and \([0.6, 0.5f_1(2.3) = 0.76, 0.6, sf_2(2.3) = 0.36]\). Another way to read Figure 6 is setting the same probability for both firm and they get different prices \([p_1 = 0.76, p_2 = 0.57, \gamma f_1^{-1}(0.5) = 1.06]\) as the strategy. Figure 6 also shows the firms charge the same price will have different profits. For example, if price \(p = 2.3\), the profit of firm-1 and firm-2 is 14.1 and 7.5, respectively. The higher capacity firm has higher probability to obtain more profits.

**Discussion for Bertrand Game**

The result of pure strategy solutions to capacity unconstraint game reveals that the price of strategy profit is \((c^*, c^*)\) and the profits for two firms are zero (Chowdhury and Sengupta, 2004). If both firms have the same capacity constraint, \(L\), the equilibrium solutions based on the quantity of capacity constraint are different. Specifically, if both firms can only supply \(0 < L < \left(\frac{1}{3}\right)M\) product, Bertrand-Nash duopoly non-cooperative pure strategy solutions can be found. The equilibrium price is \(p^* = M - 2L\) and the profit is \(\pi = (p^* - c)L\). If the product capacity is more than \(\left(\frac{1}{3}\right)M\) mixed strategy solutions can be found by the numerical tables and computer graphics.

If Bertrand price game has unequally constraint capacity, the functions of probability are not continuous because they include minimum and maximum ones. In this situation, one of the best ways to solve the problem is by using computers. The pair of probabilities and the table digits can be used to figure out various mixed strategy solutions and their correspondent profits.

**DISCUSSION AND CONCLUSION**

This paper applies different methods to identify the Cournot output and Bertrand price Nash equilibrium. By using partial derivative; the Cournot game can be solved. The example of duopoly Cournot competition reveals at the equilibrium the output strategy profile is \((q_1^*, q_2^*) = (4, 4)\) and the profit for each firm is 16. To solve duopoly Bertrand game, it is convenient by using the figure (Figure 3) to explain the logic reason if both firms have output constraint. If the output constraint \(L\) is in the range \(0 < L \leq 4\), a pure price strategy equilibrium can be obtained. Numerically, as the suggestions of Cournot game \((M = 12.1, c = 0.1)\), if the outputs \(q_1 = q_2 = 4\), the price strategy profile is \((p_1^*, p_2^*) = (4.1, 4.1)\); both firms have the profit 16. Thus the results show, at this point, i.e. \((q_1^*, q_2^*) = (4, 4), (p_1^*, p_2^*) = (4.1, 4.1), and (\pi_1, \pi_2) = (16, 16)\), the Nash equilibrium of Cournot and Bertrand game is coincident.
If the output constraint is \( L > 4 \), Bertrand game has to investigate by two situations. One is mixed strategy for equal capacity constraint and the other is mixed strategy for different capacity constraint. Because both situations have discontinuous maximum or minimum equations, using computer to solve these simultaneous equations is convenient. The use of computer technique is also easily to find out the solution of Cournot repeated competition game; it is illustrated on Figure 2.

The game theory has many variables. Cournot output and Bertrand price are the game of single variable preceding the invention of game theory. This paper basically studies the non-cooperative duopoly game. The specialty of this study is using a computer to figure out the solutions of Cournot repeated game and Bertrand discontinuous duopoly games. In the future, if complicated models are proposed, the computer can perform the simulation results and solve the problem.

**REFERENCE**


