Pricing American Call Options with Dividend and Stochastic Interest Rate

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ABSTRACT

This article presents a closed form solution for pricing American stock call options with one known dividend under the Ho-Lee stochastic interest rate assumptions, via a construction of some replicating portfolio. The correlation between the underlying stock price process and the stochastic discount factor process is suitably established. Numerical analyses demonstrate that there are some crucial parameters, the correlation coefficient between the stock price process and the discount factor process, and the amount of dividend, which have an impact on the option price. These results provide researchers and participants with some pricing applications in the real financial market.

Keywords: American stock call options, Correlation coefficient, Dividends, Stochastic interest rate

INTRODUCTION

Option pricing models were first presented by Black and Scholes (1973), and Merton (1973). It is well-known that an American call option on a stock that pays no dividends during the life of the option will not be exercised early, and hence can be evaluated as a European stock option with the standard Black-Scholes formula. If the underlying stock does pay a dividend during the life of the option, however, early exercise could possibly be an optimal choice and further investigation becomes interesting. Moreover, the pricing of American options becomes more complicated as the randomness of the interest rate becomes further involved.

American call options with given dividends, were analytically evaluated by the so called Roll-Geske-Whaley model, through evaluating compound options. The purpose of this article is to develop a closed form pricing formula for American stock call options with one given dividend, subject to Ho-Lee stochastic interest rate model. The correlation between the underlying stock price and the discount factor is explicitly expressed. And numerical analyses illustrate that the correlation between the underlying stock price and the discount factor imposes a discernible influence on the dynamics of the option price. Furthermore, the impacts of the dynamic for distinct initial stock prices, are inspected as well. These offer profitable information that can be applied in the real financial market.

The remainder of this article is organized as follows. In the next section the valuation model is established, and the required conditions and closed-form formula for finding the option pricing for the discussed American stock call option are derived. Numerical illustrations for comparative analyses, for pricing are then conducted. In the final section some conclusion are discussed.

BACKGROUND OF THE PROPOSED MODEL

Some Notations

Before introducing the proposed model, some notations employed throughout this article are stated as follows:
Suppose an economy where all the securities evolved is assumed to be governed by a stochastic discount factor, a positive stochastic process, denoted by \( \{ Q_u \} \), which is adapted to the given filtration, \( \{ A_u \} \) from a well-defined probability space \( (\Omega, A, \mu) \). For any security which pays \( A_u \) for the measurable random amount \( p_u \) at time \( u \), the price at time \( \tau < u \) is
\[
p\tau = E\left( \frac{Q_u}{Q\tau} \cdot p_u \mid A\tau \right).
\]

Let \( W_u = \frac{Q_u}{Q_0} \) and \( \mu_{W_u} = \ln B_{0,u} - \frac{1}{2} \sigma_0^2 u \), then \( W_u \) follows a log-normal distribution:
\[
\ln W_u \sim N_1(\mu_{w_u}, \sigma_0^2 u).
\]
(1)

Since there is a fixed cash dividend of \( D \) dollars at time \( t \), the underlying stock price prior to \( t \) will involve the effect of paying the dividend. To evaluate the option at time 0, the associated stock price at time 0 must be adjusted so that the dynamics of the stock price throughout its life time can be modeled. Let the stock price containing dividends be denoted by \( S' \), and the stock price without dividends be \( S' \), then \( S' = S_0 - DB_{0,t} \). In this article, the dynamic behaviors of \( Q \) and \( S \) at the time \( T \), are expressed by
\[
\left\{ \begin{array}{l}
\frac{Q_T}{Q_0} = B_{0,T} \exp\left( -\frac{1}{2} \sigma_0^2 T - \sigma_0 Z_{q_T} \right), \\
\frac{S_T}{S_0 - DB_{0,t}} = \frac{1}{B_{0,T}} \exp\left( \frac{1}{2} (2 \rho \sigma_0 \sigma_0 - \sigma_0^2 \sigma_0^2) T + \sigma_0 Z_{s_T} \right) \end{array} \right\},
\]
where \( (Z_{0\tau}, Z_{\tau})^d \sim N_2(0,0,T,T; \rho) \).

Adopting the aforementioned notation, define \( W_t = \frac{Q_T}{Q_0} \), \( \mu_{\omega} = \ln B_{0,t} - \frac{1}{2} \sigma_{0}^2 T \), and

\[
\mu_{\omega} = \ln(S_0 - DB_{0,t}) - \ln B_{0,t} + \rho \sigma_{0} \sigma_{s} T - \frac{1}{2} \sigma_{s}^2 T ,
\]

it follows that

\[
\ln(S_T, \ln W_t) \sim N_2(\mu_{\omega} + \sigma_{s}^2 T, \sigma_{0}^2 T; \rho).
\]  \(2\)

Therefore, the joint relation between \( S \) and \( Q \), after the setting of stochastic interest rate, can be established by expression (2). Here, \( \rho \) is the correlation coefficient between the two processes, \{\ln S_T\} and \{\ln W_T\}.

**THE PROPOSED MODEL**

**Replicating Portfolio for the Discussed American Stock Call Option**

For simplicity, the current time is assumed to be zero. For the underlying stock, with maturity date \( T \), only a fixed cash dividend of \( D \) dollars is paid at time \( t \). The time scale in years until the ex-dividend date is \( t \), where \( t \in (0,T) \). Moreover, the interest rate is supposed to follow the Ho-Lee model discussed in subsection 2.2. Adopting the Roll-Geske-Whaley development, a synthetic portfolio with a combination of the following stated options is proposed:

(A). A long position in a European stock call option with time to maturity \( T \) and exercise price \( K \), where the underlying asset is the same stock as the replicated American call option.

(B). A short position in a European compound stock call option with time to maturity \( t \) and exercise price \( S_t^* + D - K \), the underlying asset is the same as the European call option in portfolio (A), and \( S_t^* \) is the “unique” solution to the equation, \( C_t(S_t^*, B_t, K, T-t) = S_t^* + D - K \).

(C). A long position in a European stock call option with time to maturity \( t \) and exercise price \( S_t^* \).

Again the underlying asset is the same stock as the replicated American call option.

**Some Analytical Results**

In this subsection, the equivalence of the aforementioned synthetic portfolio, combining (A), (B) and (C), and the discussed American stock call option, will be proved analytically under some regular conditions. First, at time zero, prices for each of the European options, (A), (B) and (C), denoted by \( C_0^{(A)} \), \( C_0^{(B)} \) and \( C_0^{(C)} \), respectively, are obtained as follows:

**Theorem 1**:

\[
\begin{align*}
C_0^{(A)} &= (S_0 - DB_{0,t}) \exp(2\rho \sigma_{0} \sigma_{s} T) \Phi_1(a) - KB_{0,T} \Phi_1(1) \quad (3) \\
C_0^{(B)} &= (S_0 - DB_{0,t}) \exp(2\rho \sigma_{0} \sigma_{s} T) \Phi_2(a,b;\nu) - KB_{0,T} \Phi_2(c,d;\nu) - \left( S_t^* + D - K \right) B_{0,t} \Phi_1(d) \quad (4) \\
C_0^{(C)} &= (S_0 - DB_{0,t}) \exp(2\rho \sigma_{0} \sigma_{s} T) \Phi_1(b) - S_t^* B_{0,t} \Phi_1(d) \quad (5)
\end{align*}
\]

where \( a = \left( S_0 - DB_{0,t} \right) / \left( KB_{0,T} \right) \) and \( b = \left( 2\rho \sigma_{0} \sigma_{s} + \frac{1}{2} \sigma_{s}^2 \right) T / \left( S_t^* B_{0,t} \right) \).
c = a − σ_s \sqrt{T} , \quad d = b − σ_s \sqrt{t} , \quad and \quad v = \frac{t}{\sqrt{T}} , \quad a fixed number. \quad Again \quad S^*_t \quad is \quad the \quad unique \quad solution \quad of \quad the \quad equation, \quad C_t(S^*_t, B_{t,T}, K, T-t) = S^*_t + D - K .

Proof: A sketched of the proof is referred to the Appendix.

**Corollary 1:** At time \ t \ , \ the ex-dividend date, the price of the European option (A) is

\[ C_t^{(s)} = C_t(S^*_t, B_{t,T}, K, T-t) = \exp\left[2\rho \sigma_q \sigma_s (T-t)\right] \Phi_1(d_1) - K B_{t,T} \Phi_1(d_2), \]

where \ d_1 = \frac{1}{\sigma_s \sqrt{T-t}} \left\{ \ln \left( \frac{S^*_t}{K B_{t,T}} \right) + \left(2 \rho \sigma_q \sigma_s + \frac{1}{2} \sigma_s^2 \right)(T-t) \right\} , \quad and \quad d_2 = d_1 - \sigma_s \sqrt{T-t} .

Proof: The result immediately follows from Theorem 1.

To prove that the synthetic portfolio of the defined European stock call options (A), (B) and (C), is a replicating portfolio for the discussed American stock call option, it is sufficient to show that \ f(S_t) \ is an increasing function of \ S_t , where \ f(S_t) = (S_t - X + D) - C_t(S_t, B_{t,T}, K, T-t) .

Since \ \frac{\partial \Phi_1(d_1)}{\partial S_t} = \phi_1(d_1) \frac{\partial d_1}{\partial S_t} , \quad \frac{\partial \Phi_1(d_2)}{\partial S_t} = \phi_1(d_1) \exp \left( d_1 \sigma_s \sqrt{T-t} - \frac{1}{2} \sigma_s^2 \sqrt{T-t} \right) , \quad K B_{t,T} \phi_1(d_2) = S_t \phi_1(d_1) \exp \left[2 \rho \sigma_q \sigma_s (T-t)\right] ,

after algebra, the partial derivative, \ \frac{\partial f(S_t)}{\partial S_t} = 1 - \exp \left[2 \rho \sigma_q \sigma_s (T-t)\right] \Phi_1(d_1) , \quad is \quad obtained.

Define \ g(\rho) = 1 - \exp \left[2 \rho \sigma_q \sigma_s (T-t)\right] \Phi_1(d_1) , \quad then \quad \frac{\partial g(\rho)}{\partial \rho} = -2 \sigma_q \sqrt{(T-t)} \exp \left[2 \rho \sigma_q \sigma_s (T-t)\right] \left( \phi_1(d_1) + \frac{d_1}{\sigma_s \sqrt{T-t}} \right) < 0 .

Since \ g(0) \geq 0 , \quad therefore \quad \frac{\partial f(S_t)}{\partial S_t} \geq 0 , \quad when \ \rho \leq 0 . \quad That \quad is, \quad f(S_t) \quad is \quad an \quad increasing \quad function \quad of \quad S_t , \quad as \ \rho \leq 0 . \quad When \ \rho > 0 , \quad the \quad sign \quad of \quad \frac{\partial f(S_t)}{\partial S_t} \quad is \quad uncertain. \quad However, \quad since \quad \frac{\partial f(S_t)}{\partial S_t} \quad is \quad a \quad continuous \quad function \quad of \quad \rho \quad , \quad thus \ there \quad exists \quad \rho_* > 0 \quad , \quad function \quad of \quad \sigma_s , \quad \sigma_q , \quad \frac{S_t}{K B_{t,T}} \quad and \quad T-t , \quad such \quad that \quad \frac{\partial f(S_t)}{\partial S_t} \bigg|_{\rho=\rho_*} = 0 . \quad The \quad following \quad results \quad are \quad obtained:

**Corollary 2:** Suppose \ \rho_* \ satisfies \ the \ equation \ \exp \left[2 \rho_0 \sigma_q \sigma_s (T-t)\right] \Phi_1(d_1(\rho_*)) = 1 , \quad then \ \rho_* > 0 \quad and \quad f(S_t) \quad is \quad an \quad increasing \quad function \quad of \quad S_t , \quad for \ \rho < \rho_* .

For simplicity, we suppose that \ \rho < \rho_* ; \quad therefore \quad f(S_t) \quad is \quad an \quad increasing \quad function \quad of \quad S_t \quad in \quad the \quad following \quad discussion. \quad Furthermore \quad let \quad S^*_t \quad be \quad the \quad unique \quad solution \quad satisfying \quad f(S^*_t) = 0 , \quad that \quad is, \quad S^*_t - X + D = C_t(S^*_t, B_{t,T}, K, T-t) . \quad The \ discussed \ American \ stock \ call \ option \ shall \ be \ exercised \ if \ \quad S_t - X + D > C_t(S_t, B_{t,T}, K, T) \quad , \quad that \quad is \quad S_t > S^*_t \quad ; \quad otherwise \ \it{it \ \will \ \be \ \held \ \to \ \maturity, \ \if \ S_t - X + D \leq C_t(S_t, B_{t,T}, K, T) \quad , \quad that \quad is \quad S_t \leq S^*_t \quad .
To prove that the synthetic portfolio of the defined European stock call options (A), (B) and (C) is a replicating portfolio for the discussed American stock call option, each payoff function can be reduced to two exclusive cases expressed by cash flows.

(I). Direct Cash flows of the discussed American stock call option

At time \( t \) if \( S_t > S_t^* \), then \( f(S_t) > 0 \), and the option will be exercised immediately. The value of the discussed American stock call option is thus \( S_t - X + D \). However, if \( S_t \leq S_t^* \), then \( f(S_t) \leq 0 \), and consequently, the option will be held to maturity, with the value \( C_t(S_t, B_{t,T}, K, T - t) \).

(II). Cash flows of the discussed replicating portfolio

At time \( t \) if \( S_t > S_t^* \), then \( C_t(S_t, B_{t,T}, K, T - t) > C_t(S_t^*, B_{t,T}, K, T - t) = S_t^* - X + D \).

Therefore, the cash flow of the European compound option (B) is

\[
\max\{C_t(S_t, B_{t,T}, K, T - t) - (S_t^* - X + D), 0\} = C_t(S_t, B_{t,T}, K, T - t) - (S_t^* - X + D).
\]

Similarly, since \( \max\{S_t - S_t^*, 0\} \geq 0 \), the cash flow of the European stock call option (C) at time \( t \) is \( S_t - S_t^* \). In summary, the total cash flows of the replicating portfolio are

\[
C_t(S_t, B_{t,T}, K, T - t) - \left[ C_t(S_t, B_{t,T}, K, T - t) - (S_t^* - X + D) \right] + S_t - S_t^* = S_t - D + X.
\]

This value is exactly the same as that computed by directly analyzing the cash flows for the discussed American stock call option.

On the other hand, if \( S_t \leq S_t^* \), then \( C_t(S_t, B_{t,T}, K, T - t) \leq C_t(S_t^*, B_{t,T}, K, T - t) = S_t^* - X + D \).

Therefore, both options in (B) and (C) are zero and the total cash flows of the replicating portfolio are \( C_t(S_t, B_{t,T}, K, T - t) \). The stated cash flows at time \( t \) are summarized in Table 1.

When the condition that \( \rho < \rho_* \) holds, regardless of whether \( S_t > S_t^* \) or \( S_t \leq S_t^* \), the value of the replicating portfolio is always equal to that of the discussed American stock call option. Therefore the synthetic portfolio defining by European stock call options (A), (B) and (C) can be regarded as a replicating portfolio for the discussed American stock call option. Thus to price the discussed American stock call option, it is sufficient to evaluate the discussed European stock call options (A), (B), (C), respectively. The results are as follows:

**Theorem 2:** Suppose that \( \rho < \rho_* \), then the synthetic portfolio combining by, the European options (A), (B) and (C), is a replicating portfolio for the discussed American stock call option, with price \( C_0 \), which is \( C_0 = C_0^{(A)} - C_0^{(B)} + C_0^{(C)} \).

### Table 1: Cash flows under \( \rho < \rho_* \) assumption

<table>
<thead>
<tr>
<th>Value at time ( t )</th>
<th>( S_t &gt; S_t^* )</th>
<th>( S_t \leq S_t^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>The American call option</strong></td>
<td>( S_t - X + D )</td>
<td>( C_t(S_t, B_{t,T}, K, T - t) )</td>
</tr>
<tr>
<td><strong>Option (A)</strong></td>
<td>( C_t(S_t, B_{t,T}, K, T - t) )</td>
<td>( C_t(S_t, B_{t,T}, K, T - t) )</td>
</tr>
<tr>
<td><strong>Option (B)</strong></td>
<td>(- [C_t(S_t, B_{t,T}, K, T - t) - (S_t^* - X + D)])</td>
<td>0</td>
</tr>
<tr>
<td><strong>Option (C)</strong></td>
<td>( S_t - S_t^* )</td>
<td>0</td>
</tr>
<tr>
<td><strong>(A)+(B)+(C)</strong></td>
<td>( S_t - X + D )</td>
<td>( C_t(S_t, B_{t,T}, K, T - t) )</td>
</tr>
</tbody>
</table>
Theoretically, the existence of a replicating portfolio for the discussed American stock call option is proved. Unfortunately, the value of $\rho$, depends upon $S_t$, the stock price at time $t$, which is unknown when the current time is at time zero. Nevertheless, Theorem 2 is definitely applicable when $\rho \leq 0$. Usually, in most situations, the value of the correlation coefficient $\rho$ will be negative, thus the price of the discussed American stock call option is $C_0 = C_0^{(A)} - C_0^{(B)} + C_0^{(C)}$. The result can be stated as follows:

**Corollary 3**: When $\rho < 0$, the correlation between the stock price process and the discount factor process, is negatively correlated. The time zero price of the discussed American stock call option, with a fixed cash dividend of $D$ dollars paid at time $t$, before the maturity date $T$, under the Ho-Lee model, is then given by

$$C_0 = (S_0 - DB_{0,t}) \exp\left( 2 \rho \sigma S_0 t \right) \Phi_1(b) + \exp\left( 2 \rho \sigma S_0 T \right) \Phi_2(a, -b; -\nu) - KB_{0,t} \Phi_2(c, -d; -\nu) - (K - D) B_{0,t} \Phi_1(d) \right).$$

(6)

**Proof.** The result immediately follows from Theorem 2.

The most significant difference between the Roll-Geske-Whaley model and the proposed model lies in the assumption of random interest rate that is carried out by the Ho-Lee model. In addition the correlation between the discount factor and the underlying stock price is established via the correlation coefficient $\rho$. Without the stochastic interest rate assumption, we may set $\sigma_q = 0$, (or $\rho = 0$), then the result of Corollary 3 will be simply reduced to that of the Roll-Geske-Whaley model.

**NUMERICAL ILLUSTRATIONS**

Static analyses of the closed-form solutions for option pricing are offered in this section with the goal of offering detailed insight into their sensitivity to the some various parameter settings. A set of parameters, called the **base case**, is: $D = 1.5$, $K = 25$, $t = 0.25$, $r = 0.15$, $B_{0,t} = \exp(-ru)$, $T = 0.5$, $\sigma S = 0.5$, and $\sigma q = 0.01$. Three distinct initial stock values, $S_0 = [30, 25, 20]$, are set for call options whose statue stays in-the-money, at-the-money, and out-of-the-money, respectively.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$S_t/K$</th>
<th>0.6</th>
<th>0.8</th>
<th>1.0</th>
<th>1.4</th>
<th>1.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.8</td>
<td></td>
<td>0.982</td>
<td>0.825</td>
<td>0.518</td>
<td>0.100</td>
<td>0.014</td>
</tr>
<tr>
<td>-0.6</td>
<td></td>
<td>0.981</td>
<td>0.824</td>
<td>0.516</td>
<td>0.098</td>
<td>0.013</td>
</tr>
<tr>
<td>-0.4</td>
<td></td>
<td>0.981</td>
<td>0.823</td>
<td>0.514</td>
<td>0.096</td>
<td>0.012</td>
</tr>
<tr>
<td>-0.2</td>
<td></td>
<td>0.981</td>
<td>0.822</td>
<td>0.512</td>
<td>0.095</td>
<td>0.011</td>
</tr>
<tr>
<td>0</td>
<td></td>
<td>0.981</td>
<td>0.821</td>
<td>0.510</td>
<td>0.093</td>
<td>0.010</td>
</tr>
<tr>
<td>0.2</td>
<td></td>
<td>0.981</td>
<td>0.819</td>
<td>0.508</td>
<td>0.092</td>
<td>0.009</td>
</tr>
<tr>
<td>0.4</td>
<td></td>
<td>0.980</td>
<td>0.818</td>
<td>0.506</td>
<td>0.090</td>
<td>0.008</td>
</tr>
<tr>
<td>0.6</td>
<td></td>
<td>0.980</td>
<td>0.817</td>
<td>0.504</td>
<td>0.089</td>
<td>0.007</td>
</tr>
<tr>
<td>0.8</td>
<td></td>
<td>0.980</td>
<td>0.816</td>
<td>0.502</td>
<td>0.087</td>
<td>0.006</td>
</tr>
</tbody>
</table>

Table 2: Increasing property of $f(S_t)$; values of $\frac{\partial f(S_t)}{\partial S_t}$

Notes: All the entries are positive, that means under the conditions, $S_t/K \leq 1.8$ and $\rho \leq 0.8$, $f(S_t)$ is an increasing function of $S_t$. 
It is worthy to note that the sufficient condition, \( \rho < \rho_* \), stated in Theorem 2, is useful to theoretically verify that \( f(S_t) \) is an increasing function of \( S_t \), and furthermore to guarantee that the synthetic portfolio of the defined options (A), (B) and (C), is a replicating portfolio for the discussed American stock call option. Actually, \( \frac{\partial f(S_t)}{\partial S} \) is a decreasing function of either \( S_t/K \) or \( \rho \). For numerical demonstrations, instead of simply computing values of \( \rho_* \) in terms of \( S_t/K \), under the discussed base case, the values of \( \frac{\partial f(S_t)}{\partial S} \) are tabulated, according to different values of \( S_t/K \) and \( \rho \), to examine the increasing phenomena of \( f(S_t) \). The results listed in Table 2 show that as long as the stock price at time \( t \) is a moderate in-the-money situation, say, \( S_t/K \leq 1.8 \), also \( \rho \leq 0.8 \), the increasing property of \( f(S_t) \) in \( S_t \), is ensured; furthermore, the price of the discussed American stock call option defined by equation (6), could be straightforwardly applied.

The following numerical illustrations are performed under the restriction, that \( S_t/K \leq 1.8 \). The variation in the American stock call option prices under various amounts of cash dividends and correlation coefficients with different stock values is illustrated in Figure 1. At all option-value statues, the option price decreases as the cash dividend increases; for a fixed correlation the difference is about 1 dollar when the amount of the cash dividend ranges from 0.5 to 5.5. This is because a larger cash dividend causes a lower post-dividend stock price, thereby resulting in a diminishing call option price. On the other hand, the option price increases as the correlation coefficient increases. This agrees with the commonly accepted economic principle that a boom in the stock market goes along with a sag in the short-term interest rate market. The impact of the correlation, however, is not as dominant as the cash dividend; for a fixed amount of cash dividends, the extreme variation is around 0.1 dollar when the correlation ranges from -0.8 to 0.8.

Next, attentions are focused on the option price difference, between the proposed model, \( C_0 \), and the Roll-Geske-Whaley model, \( \tilde{C}_0 \). The parameter \( \rho \), correlation coefficient between the stock price and the discount factor, plays an important role in the comparisons. The percentage price difference is defined by \( dC = \left[ \frac{C_0 - \tilde{C}_0}{\tilde{C}_0} \right] \times 100\% \).

![Figure 1: Price dynamics for different cash dividend and correlation at distinct stock values](image-url)
The variation of the option price difference under various correlation coefficients and values is illustrated in Figure 2. For all option-value statues, the option price difference increases as the correlation coefficient increases; in fact, the price difference is negative under a negative correlation coefficient, and it turns positive whenever the stock price and the discount factor are positively correlated. This means that when the stock price and discount factor are negatively correlated, the theoretical option price should be lower than the one computed from Roll-Geske-Whaley model; on the other hand, when they are positively correlated, the theoretical option price may be underestimated by using the Roll-Geske-Whaley model. In addition, the largest price difference occurs when the stock price and the discount factor are highly correlated, reaching an absolute amount of 3%.

The sensitivity of the option price difference with respect to the correlation coefficient, as illustrated in Figure 2, also relies on the state of the initial stock price. When the initial stock price is low, the price difference of the out-of-the-money American stock call option is more sensitive to the variation of the correlation coefficient; in contrast, the sensitivity of the option price difference with respect to the change of correlation coefficient is smaller when the option is at-the-money or in-the-money. This is because the option is of the American type and so would be exercised early if the stock price is high enough. Nonetheless, when the American option is out-of-the-money at present, it should be held, to wait for a more benign evolution of the stock price. Since the out-of-the-money option depends on the uncertainty of future market behavior, the correlation coefficient between the stock price and the discount factor imposes a rather heavier impact on it.

![Figure 2: Price difference analysis](image.png)

**CONCLUSIONS**

In this article, an analytical formula for evaluating American stock call options with a given dividend under stochastic interest rate is derived. By fabricating the proper correlation between the underlying stock price process and the discount factor process, we can construct an equivalent replicating portfolio under suitable conditions, and the closed-form solutions for option price can derived analytically. As indicated by the numerical illustrations, the discussed American stock call option is sensitive to variations in the cash dividend amount and the introduced correlation coefficient. The dynamics of the call option value varies for different stock prices, is most influenced by changes in the cash dividend amounts. Numerical analyses show that a positive correlation brings a higher option price, and a
negative correlation leads to a lower one. The presented numerical results should help researchers and participants be better informed and make accurate decisions for dealing with this specific American stock call option in the real financial market.

ACKNOWLEDGEMENT

The author wishes to thank Mr. Yu-Chung Liu for his help in some numerical computations.

APPENDIX

Proof of Theorem 1, mainly depends upon the following stated result (proof is omitted).

Lemma 1: Let \( Y \) be log-normal distributed, say \( \ln Y \sim \mathcal{N}_1(\mu_Y, \sigma^2_Y) \), define \( \zeta = \mu_Y + p \sigma^2_Y / 2 \), and
\[
\psi = \mu_Y + p \sigma^2_Y, \quad \text{with} \quad p > 0, \quad \text{then for any} \quad q, r \in \mathbb{R}, \quad \text{and} \quad I_{r \geq 1} \quad \text{is an indicator function,}
\]
\[
E[Y^p I_{\{qY > r\}} \Phi_1(q \ln Y + r)] = \exp(p \zeta) \Phi_2 \left( \frac{\psi - \ln s}{\sigma_Y}, \frac{q \psi + r}{\sqrt{1 + q^2 \sigma_Y^2}}, \frac{q \sigma_Y}{\sqrt{1 + q^2 \sigma_Y^2}} \right).
\]

Proof of Theorem 1: Let \( \lambda_{sy} = \sigma_y \sqrt{1 - \rho^2}, \gamma = \alpha / \lambda_{sy}^2, \eta = (\mu_{sy} - a \mu_{WY} - \ln K) / \lambda_{sy} \),
\[
\theta = \lambda_{sy} + \eta, \quad \beta = \mu_{sy} - a \mu_{WY} + \lambda_{sy}^2 / 2, \quad \text{then by Lemma 1,}
\]
\[
C_{0}^{(A)} = E[W_T \max(S_T - K, 0)] = E_{W_T}[W_T E_{S_T|W_T} \max(S_T - K, 0) | W_T] = E_{W_T}[W_T \exp(\alpha \ln W_T + \beta) \Phi_1(\gamma \ln W_T + \eta)] - KE_{W_T}[W_T \Phi_1(\gamma \ln W_T + \eta)].
\]
\[
E_{W_T}[W_T \exp(\alpha \ln W_T + \beta) \Phi_1(\gamma \ln W_T + \theta)] = \exp(\beta E_{W_T}[W_T^{\alpha + 1} \Phi_1(\gamma \ln W_T + \theta)]
\]
\[
= \exp \left[ \beta + (\alpha + 1) \mu_{W_T} + \frac{(\alpha + 1)^2 \sigma^2_{\omega} T}{2} \right] \Phi_1 \left( \frac{\gamma(\mu_{W_T} + (\alpha + 1) \sigma^2_{\omega} T) + \theta}{\sqrt{1 + \gamma^2 \sigma^2_{\omega} T}} \right)
\]
\[
= S_T \exp(2 \rho \sigma_s \sigma_{\omega} T) \Phi_1(a) = (S_T - DB_{0,T}) \exp(2 \rho \sigma_s \sigma_{\omega} T) \Phi_1(a).
\]
\[
E_{W_T}[W_T \Phi_1(\gamma \ln W_T + \eta)] = \exp \left( \mu_{W_T} + \frac{\sigma^2_{\omega} T}{2} \right) \Phi_1 \left( \frac{\gamma(\mu_{W_T} + \sigma^2_{\omega} T) + \eta}{\sqrt{1 + \gamma^2 \sigma^2_{\omega} T}} \right) = B_{0,T} \Phi_1(c).
\]
Therefore, \( C_{0}^{(A)} = (S_T - DB_{0,T}) \exp(2 \rho \sigma_s \sigma_{\omega} T) \Phi_1(a) - KB_{0,T} \Phi_1(c). \) Similarly, results of \( C_{0}^{B} \) and \( C_{0}^{C} \) are obtained.

REFERENCES


